Graphs

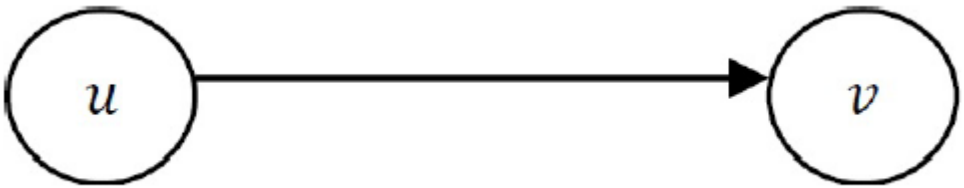
Introduction

- In the real world, many problems are represented in terms of objects and connections between them.
  - For example, in an airline route map, we might be interested in questions like: “What’s the fastest way to go from Hyderabad to New York?” or “What is the cheapest way to go from Hyderabad to New York?” To answer these questions we need information about connections (airline routes) between objects (towns). Graphs are data structures used for solving these kinds`of problems.

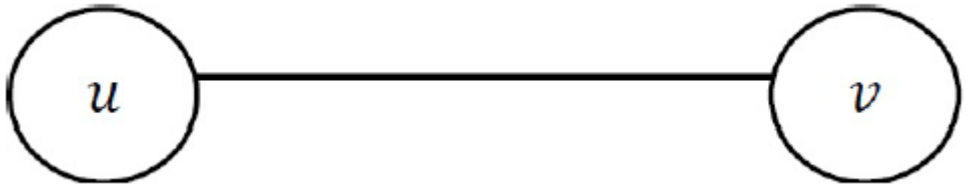
Glossary

- Graph: A graph is a pair (V, E), where V is a set of nodes, called vertices, and £ is a collection of pairs of vertices, called edges.
  - Vertices and edges are positions and store elements.

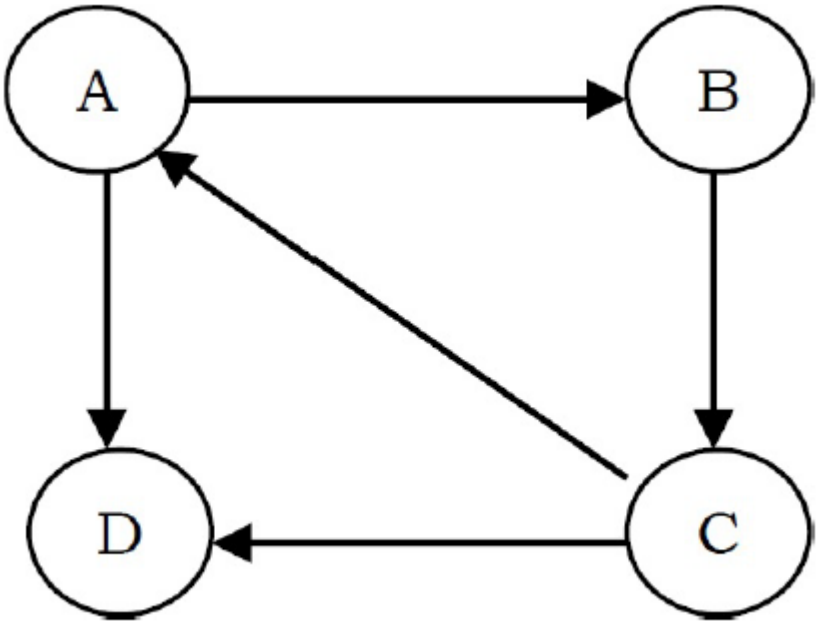
– Definitions that we use:
– ○ Directed edge:
  - ordered pair of vertices (u, v)
  - first vertex u is the origin
  - second vertex v is the destination
  - Example: one-way road traffic



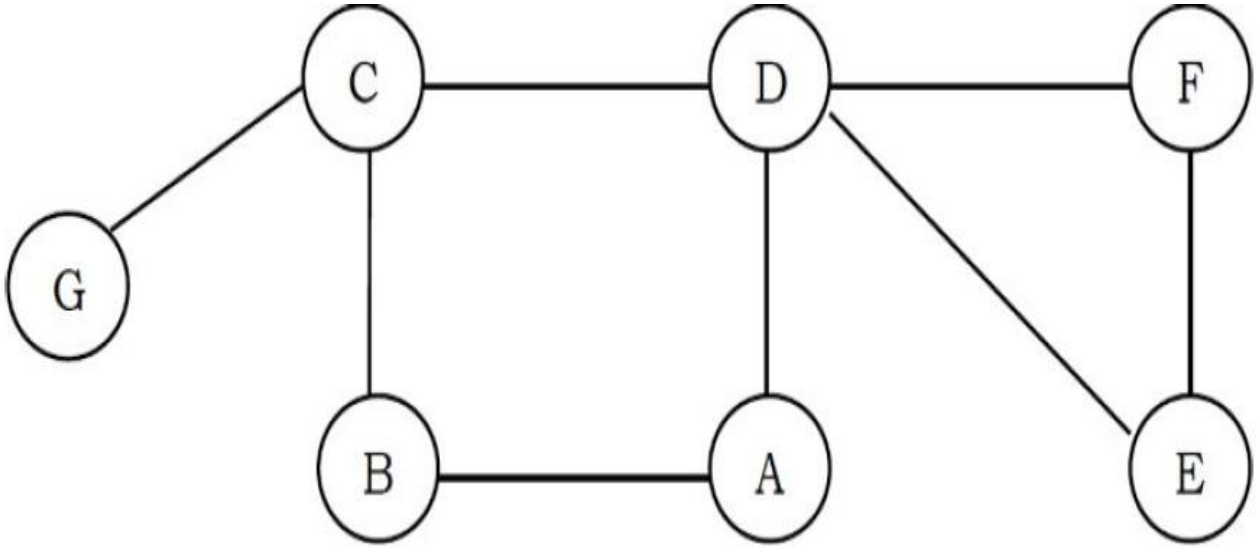
- *Undirected edge:*
  - unordered pair of vertices (*u, v*)
  - Example: railway lines.



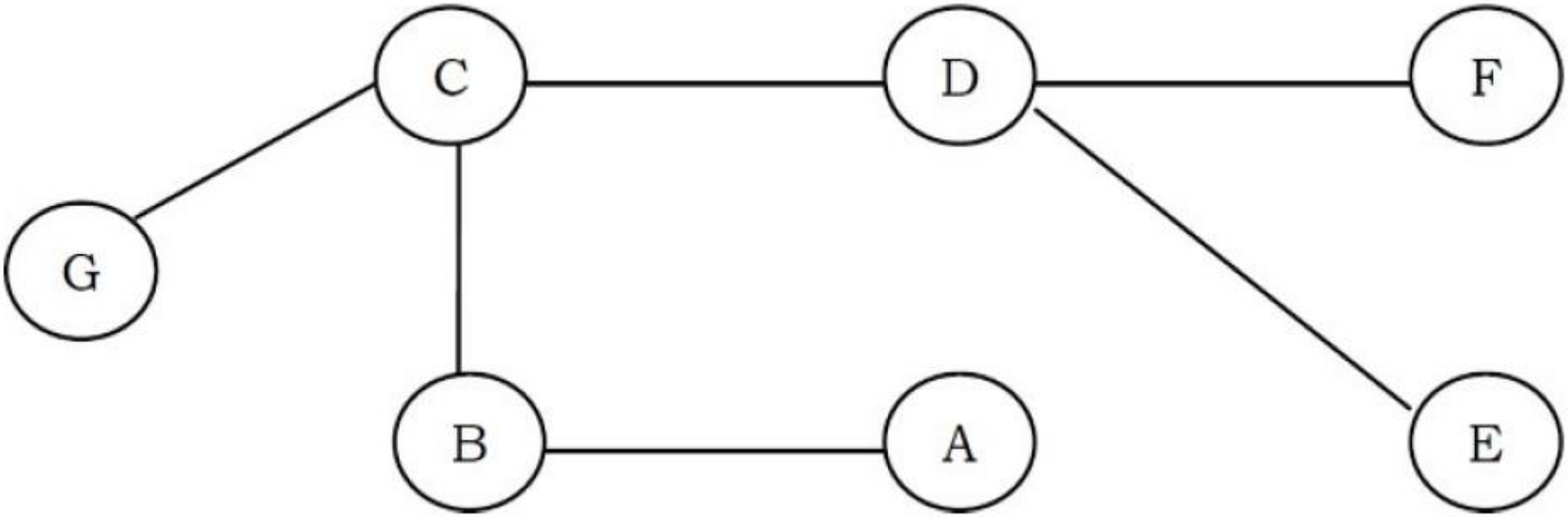
- *Directed graph:*
  - all the edges are directed
  - Example: route network



- *Undirected graph:*
- ▪ all the edges are undirected
- ▪ Example: flight network



- When an edge connects two vertices, the vertices are said to be adjacent to each other and the edge is incident on both vertices.
- A graph with no cycles is called a *tree*. A tree is an acyclic connected graph.



- A self loop is an edge that connects a vertex to itself.

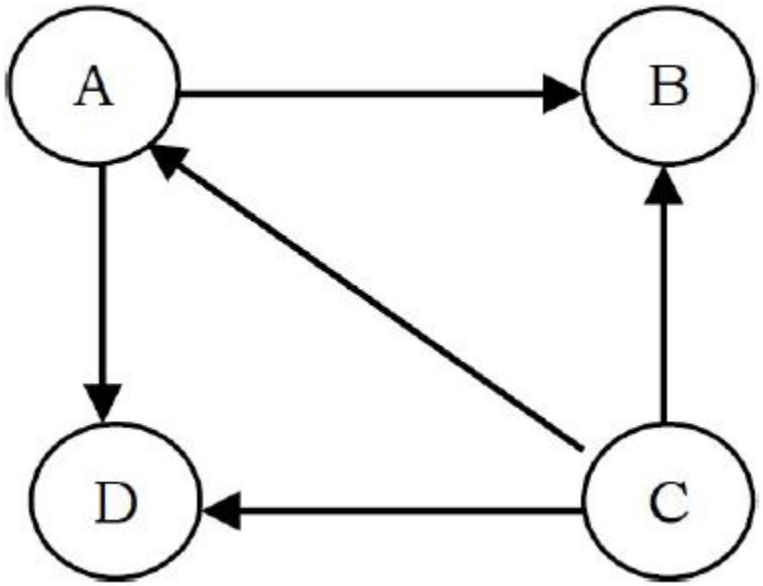
- Two edges are parallel if they connect the same pair of vertices.

- The *D*egree of a vertex is the number of edges incident on it.
- • A subgraph is a subset of a graph's edges (with associated vertices) that form a
- graph.
- • A path in a graph is a sequence of adjacent vertices. *S*imple path is a path with no
- repeated vertices. In the graph below, the dotted lines represent a path from *G* to *E.*

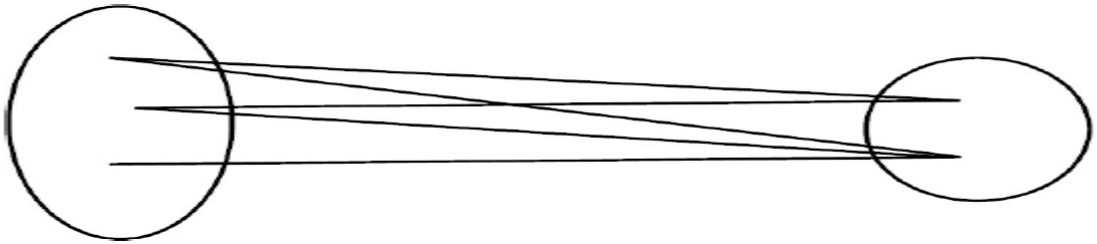
- A cycle is a path where the first and last vertices are the same. A simple cycle is a
- cycle with no repeated vertices or edges (except the first and last vertices).

- • We say that one vertex is connected to another if there is a path that contains both of
- them.
- • A graph is connected if there is a path from *every* vertex to every other vertex.
- • If a graph is not connected then it consists of a set of connected components.

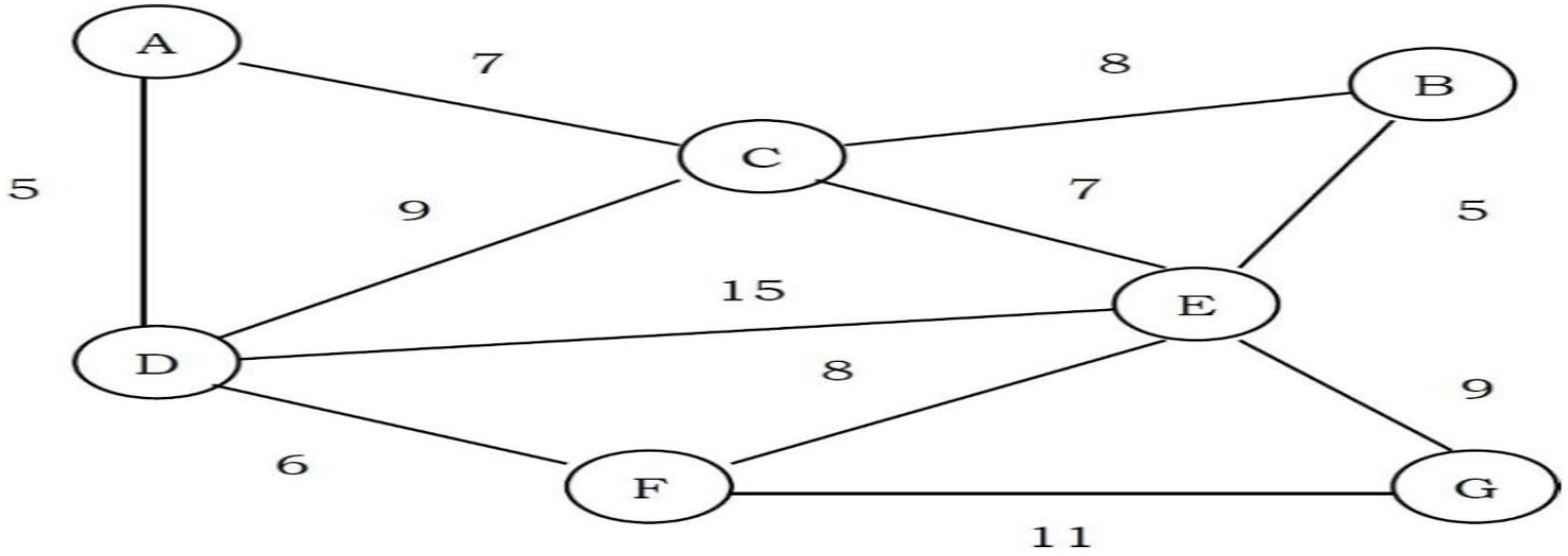
- A *directed* acyclic graph [DAG] is a directed graph with no cycles.



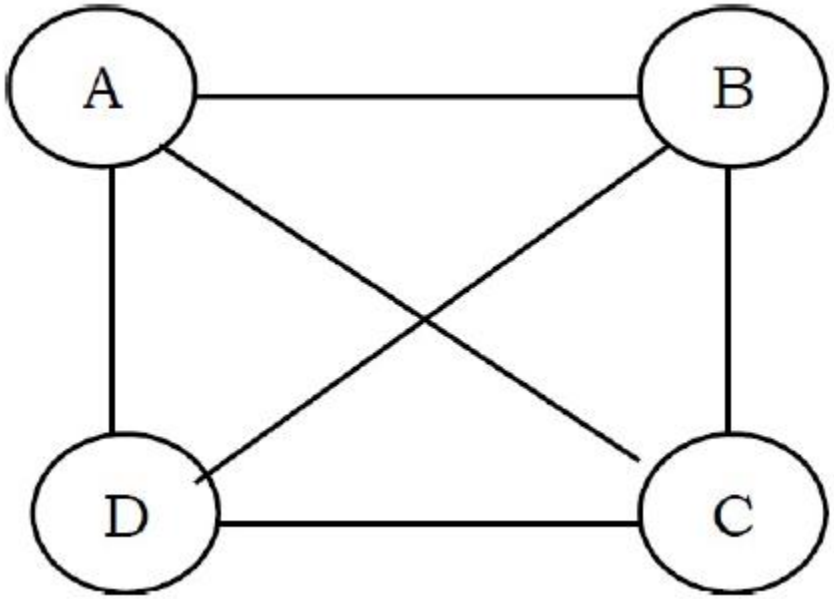
- A forest is a disjoint set of trees.
- • A spanning tree of a connected graph is a subgraph that contains all of that graph's vertices and is a single tree. A spanning forest of a graph is the union of spanning trees of its connected components.
- • A bipartite graph is a graph whose vertices can be divided into two sets such that all edges connect a vertex in one set with a vertex in the other set.



- In *weighted graphs* integers (*weights*) are assigned to each edge to represent (distances or costs).



- Graphs with all edges present are called *complete* graphs.



- Graphs with relatively few edges (generally if it edges < $|V|$ log $|V|$) are called *sparse graphs.*
  - • Graphs with relatively few of the possible edges missing are called *dense.*
  - • Directed weighted graphs are sometimes called *network.*
  - • We will denote the number of vertices in a given graph by $|V|$, and the number of edges by $|E|$. Note that E can range anywhere from 0 to $|V|(|V| - 1)/2$ (in undirected graph). This is because each node can connect to every other node.

Applications of Graphs

- Representing relationships between components in electronic circuits
  - Transportation networks: Highway network, Flight network
  - Computer networks: Local area network, Internet, Web
  - Databases: For representing ER (Entity Relationship) diagrams in databases, for representing dependency of tables in databases

Graph Representation

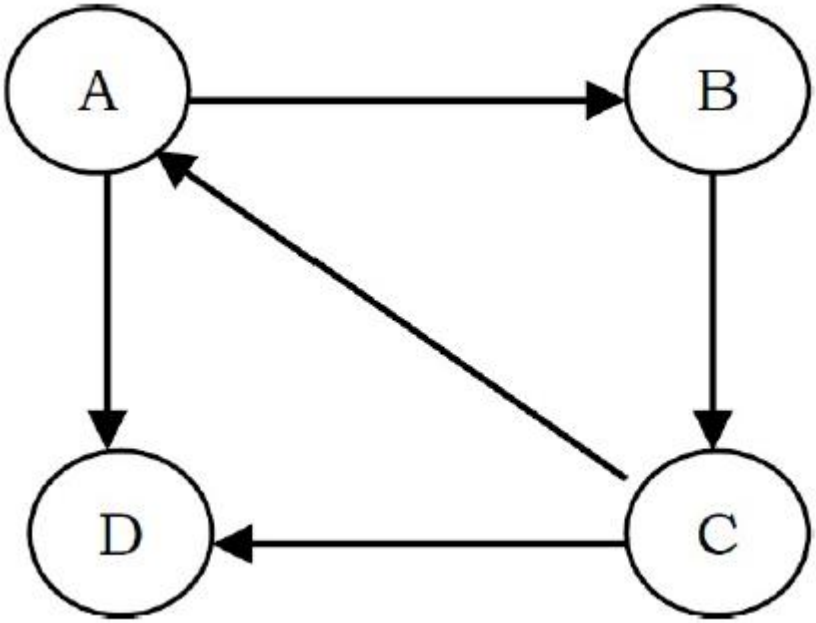
- to manipulate graphs we need to represent them in some useful form.
- Basically, there are three ways of doing this:
  - Adjacency Matrix
  - Adjacency List
  - Adjacency Set

- **Adjacency Matrix**
- **Graph Declaration for Adjacency Matrix**
- First, let us look at the components of the graph data structure. To represent graphs, we need the
- number of vertices, the number of edges and also their interconnections. So, the graph can be
- declared as:

```
struct Graph {
  int V;
  int E;
  int **Adj; //Since we need two dimensional matrix
};
```

- **Description**
- In this method, we use a matrix with size $V \times V$. The values of matrix are boolean. Let us assume the matrix is *Adj*. The value *Adj*[*u, v*] is set to 1 if there is an edge from vertex u to vertex v and 0 otherwise.
- In the matrix, each edge is represented by two bits for undirected graphs. That means, an edge from **u** to **v** is represented by 1 value in both *Adj*[**u**,**v**] and *Adj*[*u,v*]. To save time, we can process only half of this symmetric matrix. Also, we can assume that there is an "edge" from each vertex to itself. So, *Adj*[u, u] is set to 1 for all vertices.

- If the graph is a directed graph then we need to mark only one entry in the adjacency matrix. As an example, consider the directed graph below.



- The adjacency matrix for this graph can be given as:

	A	B	C	D
A	0	1	0	1
B	0	0	1	0
C	1	0	0	1
D	0	0	0	0

- Now, let us concentrate on the implementation. To read a graph, one way is to first read the vertex names and then read pairs of vertex names (edges). The code below reads an undirected graph.

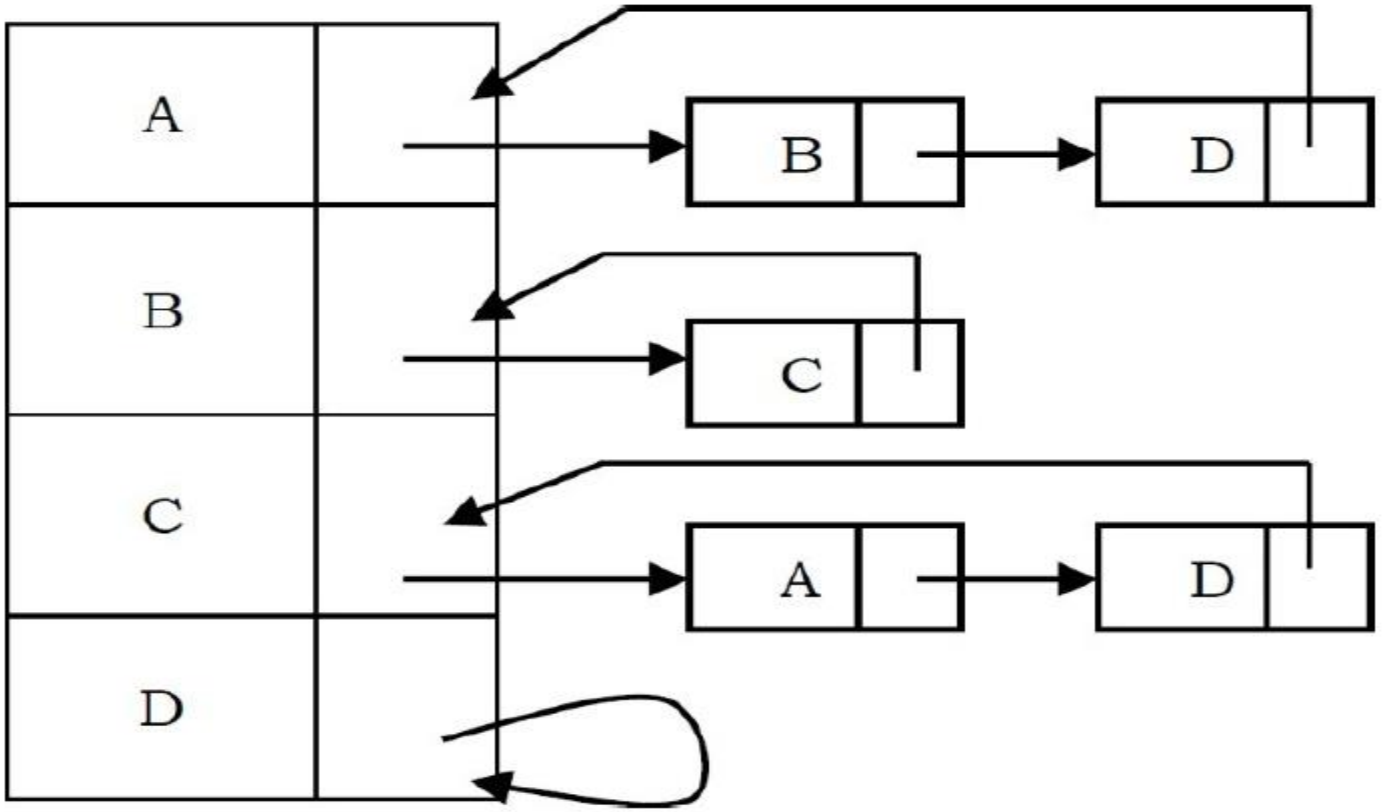
```
//This code creates a graph with adj matrix representation
struct Graph *adjMatrixOfGraph() {
    int i, u, v;
    struct Graph *G = (struct Graph *) malloc(sizeof(struct Graph));
    if(!G) {
        printf("Memory Error");
        return;
    }
    scanf("Number of Vertices: %d, Number of Edges:%d", &G→V, &G→E);
    G→Adj = malloc(sizeof(G→V  * G→V));
    for(u = 0; u < G→V; u++)
            for(v = 0; v < G→V; v++)
                    G→Adj[v][v] = 0;
    for(i = 0;  i < G→E; i++) {
        //Read an edge
        scanf("Reading Edge: %d %d", &u, &v);
        //For undirected graphs set both the bits
        G→ Adj[u][v] = 1;
        G→ Adj[v][u] = 1;
    }
    return G;
}
```

- The adjacency matrix representation is good if the graphs are dense. The matrix requires O(V2) bits of storage and O(V2) time for initialization. If the number of edges is proportional to V2, then there is no problem because V2 steps are required to read the edges. If the graph is sparse, the initialization of the matrix dominates the running time of the algorithm as it takes takes O(V2).

- **Adjacency List**
- **Graph Declaration for Adjacency List**
- In this representation all the vertices connected to a vertex *v* are listed on an adjacency list for that vertex *v*. This can be easily implemented with linked lists. That means, for each vertex *v* we use a linked list and list nodes represents the connections between *v* and other vertices to which *v* has an edge.
- The total number of linked lists is equal to the number of vertices in the graph. The graph ADT can be declared as:

```
struct Graph {
    int V;
    int E;
    int *Adj; //head pointers to linked list
};
```

- **Description**
- Considering the same example as that of the adjacency matrix, the adjacency list representation can be given as:



- Since vertex A has an edge for B and D, we have added them in the adjacency list for A. The same is the case with other vertices as well.

```
//Nodes of the Linked List
struct ListNode {
    int vertexNumber;
    struct ListNode *next;
}
```

```
//This code creates a graph with adj list representation
struct Graph *adjListOfGraph() {
    int i, x, y;
    struct ListNode *temp;
    struct Graph *G = (struct Graph *) malloc(sizeof(struct Graph));
    if(!G) {
        printf("Memory Error");
        return;
    }
```

```
scanf("Number of Vertices: %d, Number of Edges:%d", &G→V, &G→E);
G→Adj = malloc(G→V * sizeof(struct ListNode));

for(i = 0; i  < G→V; i++) {
    G→Adj[i] = (struct ListNode *) malloc(sizeof(struct ListNode));
    G→Adj[i]→vertexNumber = i;
    G→Adj[i]→next = G→ Adj[i];
}
```

```
    for(i = 0;  i < E; i++) {
        //Read an edge
        scanf("Reading Edge: %d %d", &x, &y);
        temp = (struct ListNode *) malloc(struct ListNode);
        temp→vertexNumber = y;
        temp→next = G→Adj[x];
        G→ Adj[x]→next = temp;
        temp = (struct ListNode *) malloc(struct ListNode);
        temp→vertexNumber = y;
        temp→next = G→ Adj[y];
        G→Adj[y]→ next= temp;
    }

    retutn G;
}
```

- For this representation, the order of edges in the input is *important*. This is because they determine the order of the vertices on the adjacency lists.
- The same graph can be represented in many different ways in an adjacency list. The order in which edges appear on the adjacency list affects the order in which edges are processed by algorithms.

- **Disadvantages of Adjacency Lists**
- Using adjacency list representation we cannot perform some operations efficiently. As an example, consider the case of deleting a node. . In adjacency list representation, it is not enough if we simply delete a node from the list representation, if we delete a node from the adjacency list then that is enough.
- For each node on the adjacency list of that node specifies another vertex. We need to search other nodes linked list also for deleting it. This problem can be solved by linking the two list nodes that correspond to a particular edge and making the adjacency lists doubly linked. But all these extra links are risky to process.

- **Adjacency Set**
- It is very much similar to adjacency list but instead of using Linked lists, Disjoint Sets [Union- Find] are used.

- **Comparison of Graph Representations:**
- Directed and undirected graphs are represented with the same structures.
- For directed graphs,everything is the same, except that each edge is represented just once. An edge from *x* to *y* is represented by a 1 value in *Agj*[*x*][*y*] in the adjacency matrix, or by adding y on *x's* adjacency list.
- For weighted graphs, everything is the same, except fill the adjacency matrix with weights instead of boolean values.

Representation	Space	Checking edge between v and w?	Iterate over edges incident to v?
List of edges	E	E	E
Adj Matrix	V^2	1	V
Adj List	$E+V$	$Degree(v)$	$Degree(v)$
Adj Set	$E+V$	$log(Degree(v))$	$Degree(v)$

Graph Traversals

- To solve problems on graphs, we need a mechanism for traversing the graphs. Graph traversal algorithms are also called *graph search* algorithms. Like trees traversal algorithms (Inorder, Preorder, Postorder and Level-Order traversals), graph search algorithms can be thought of as starting at some source vertex in a graph and "searching" the graph by going through the edges and marking the vertices. Now, we will discuss two such algorithms for traversing the graphs.
  - Depth First Search [DFS]
  - Breadth First Search [BFS]

- **Depth First Search [DFS]:**
- DFS algorithm works in a manner similar to preorder traversal of the trees. Like preorder traversal, internally this algorithm also uses stack.

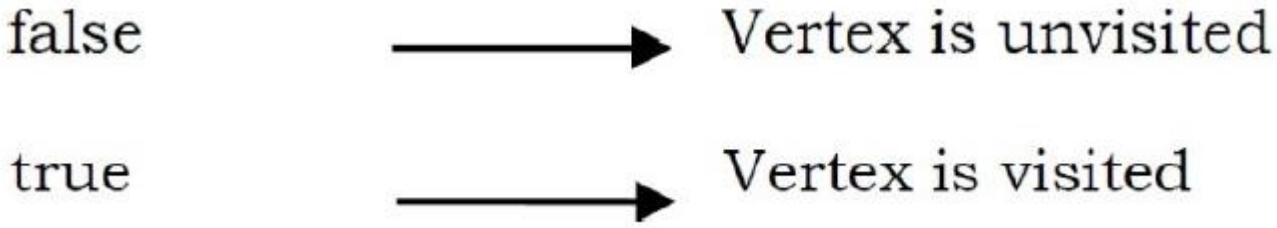
- Let us consider the following example. Suppose a person is trapped inside a maze. To come out from that maze, the person visits each path and each intersection (in the worst case). Let us say the person uses two colors of paint to mark the intersections already passed. When discovering a new intersection, it is marked grey, and he continues to go deeper.

- After reaching a “dead end” the person knows that there is no more unexplored path from the grey intersection, which now is completed, and he marks it with black. This “dead end” is either an intersection which has already been marked grey or black, or simply a path that does not lead to an intersection

- The intersections of the maze are the vertices and the paths between the intersections are the edges of the graph. The process of returning from the “dead end” is called *backtracking*. We are trying to go away from the starting vertex into the graph as deep as possible, until we have to backtrack to the preceding grey vertex. In DFS algorithm, we encounter the following types of edges.

Tree edge: encounter new vertex
Back edge: from descendent to ancestor
Forward edge: from ancestor to descendent
Cross edge: between a tree or subtrees

- For most algorithms boolean classification, unvisited/visited is enough (for three color implementation refer to problems section). That means, for some problems we need to use three colors, but for our discussion two colors are enough.



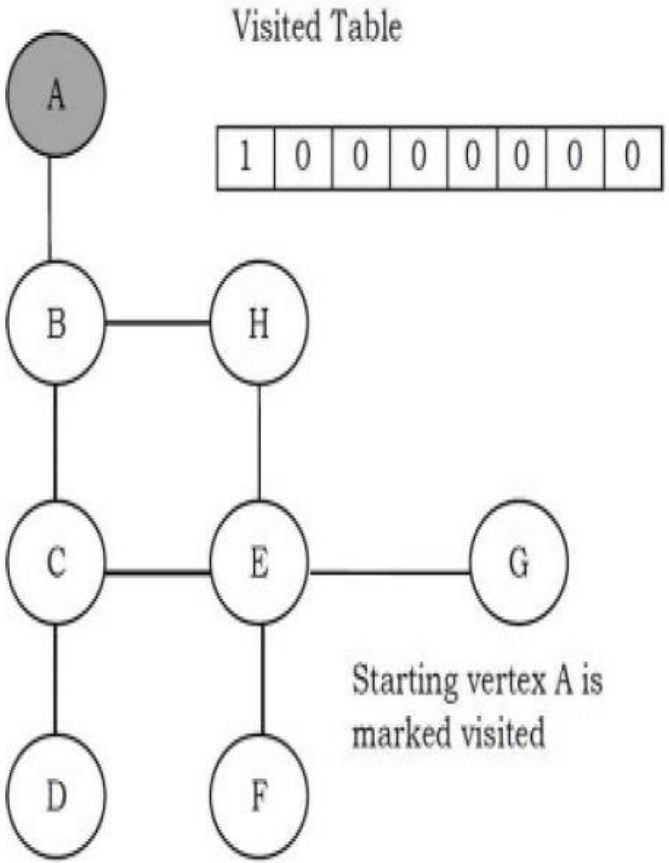
- Initially all vertices are marked unvisited (false). The DFS algorithm starts at a vertex *u* in the graph. By starting at vertex *u* it considers the edges from *u* to other vertices.
- If the edge leads to an already visited vertex, then backtrack to current vertex *u*. If an edge leads to an unvisited vertex, then go to that vertex and start processing from that vertex.
- That means the new vertex becomes the current vertex. Follow this process until we reach the dead-end. At this point start *backtracking.*

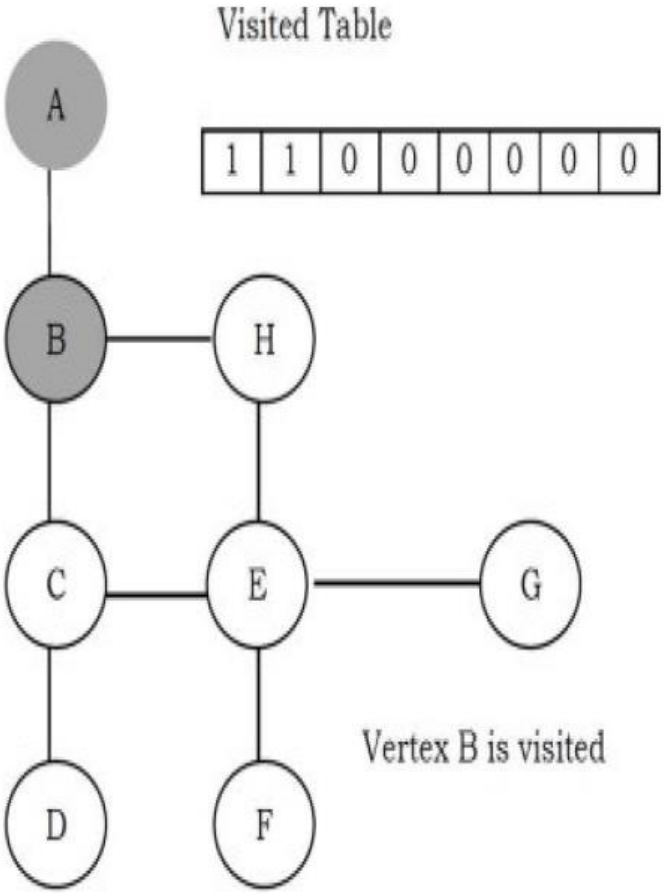
- The process terminates when backtracking leads back to the start vertex. The algorithm based on this mechanism is given below: assume Visited[] is a global array.

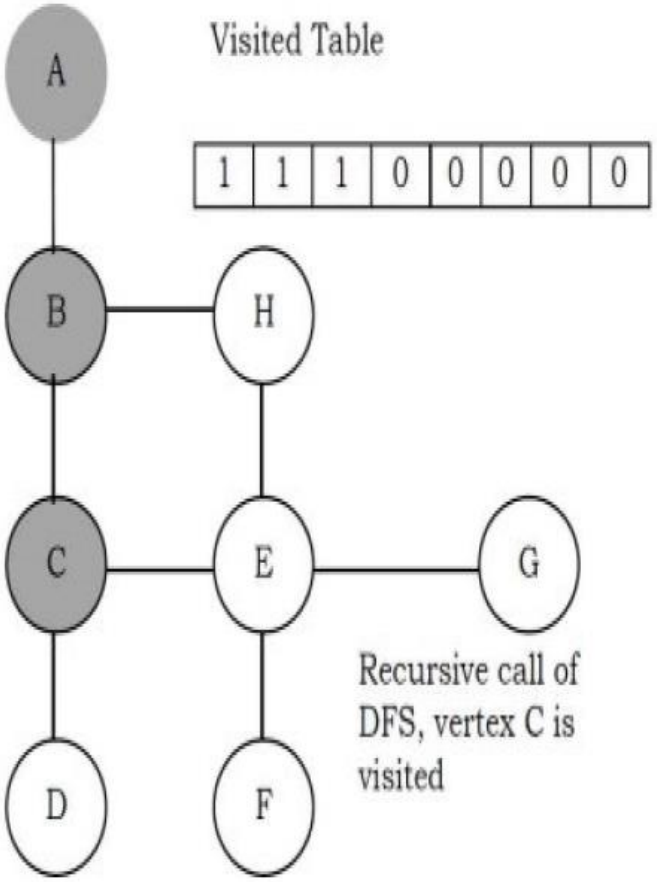
```
int Visited[G→V];
void DFS(struct Graph *G, int u) {
    Visited[u] = 1;
    for( int v = 0; v < G→V; v++ ) {

        /* For example, if the adjacency matrix is used for representing the
           graph, then the condition to be used for finding unvisited adjacent
           vertex of u  is: if( !Visited[v] && G→Adj[u][v] ) */

        for each unvisited adjacent node v of u {
            DFS(G, v);
        }
    }
}
```

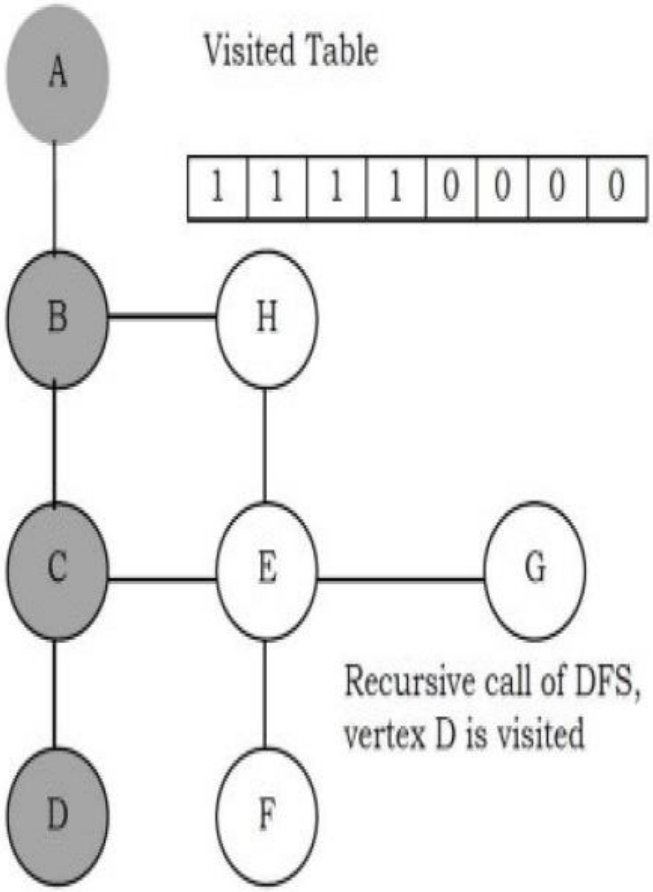
```
void DFSTraversal(struct Graph *G) {
    for (int i = 0; i< G→V;i++)
        Visited[i]=0;

    //This loop is required if the graph has more than one component
    for (int i = 0; i< G→V;i++)
        if(!Visited[i])
            DFS(G, i);
}
```

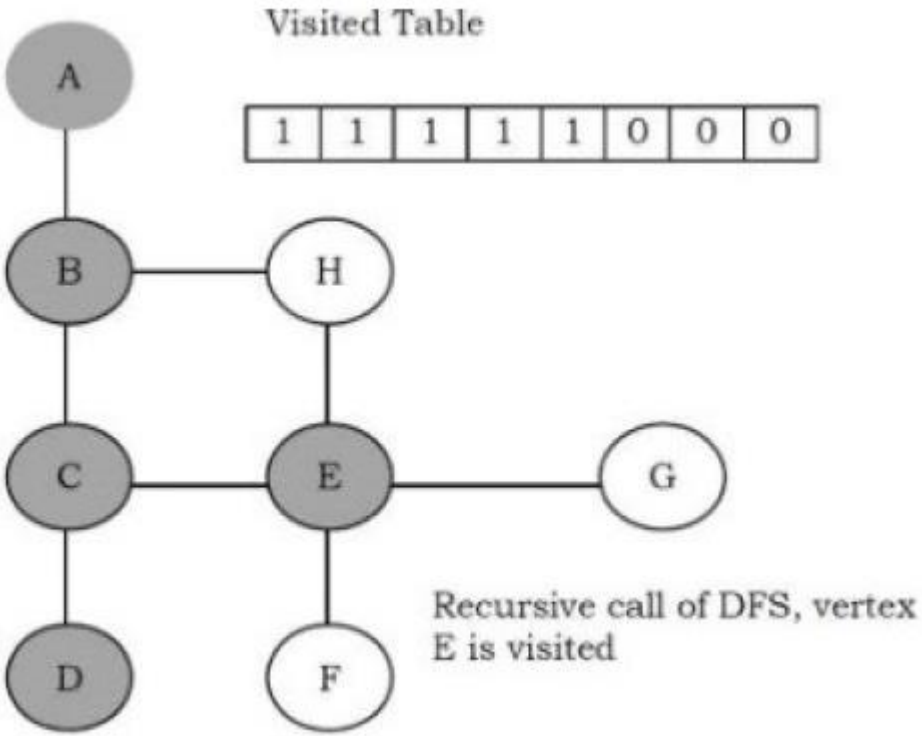
- As an example, consider the following graph. We can see that sometimes an edge leads to an already discovered vertex. These edges are called *back edges*, and the other edges are called *tree edges* because deleting the back edges from the graph generates a tree.
- The final generated tree is called the DFS tree and the order in which the vertices are processed is called *DFS numbers* of the vertices. In the graph below, the gray color indicates that the vertex is visited (there is no other significance). We need to see when the Visited table is updated.

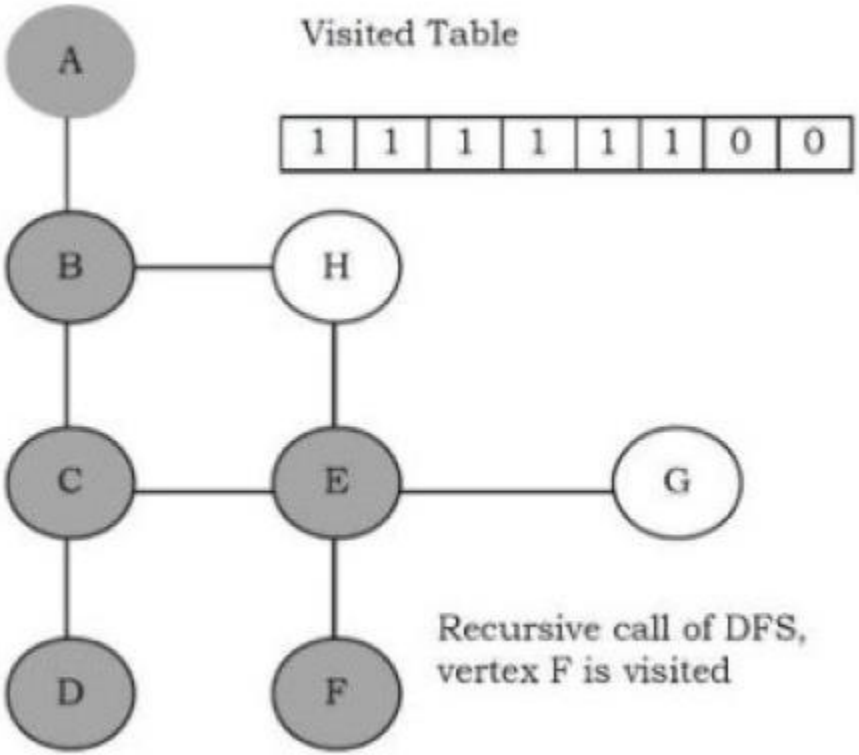

Visited Table
A
1 0 0 0 0 0 0 0
B
H
C
E
G
D
F
Starting vertex A is
marked visited

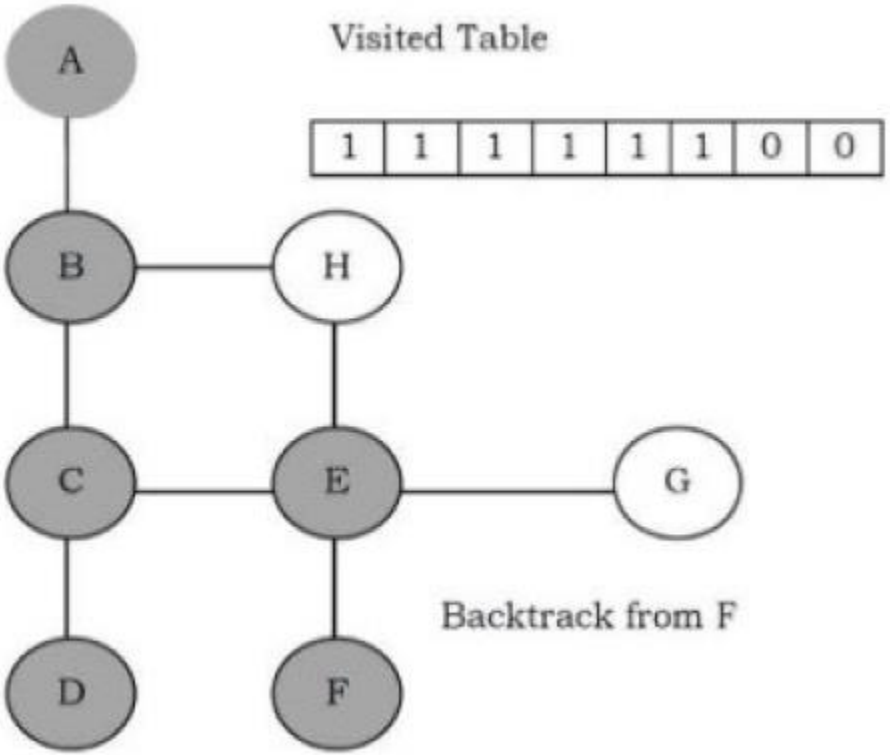

Visited Table
A
1 1 0 0 0 0 0 0
B
H
C
E
G
Vertex B is visited
D
F

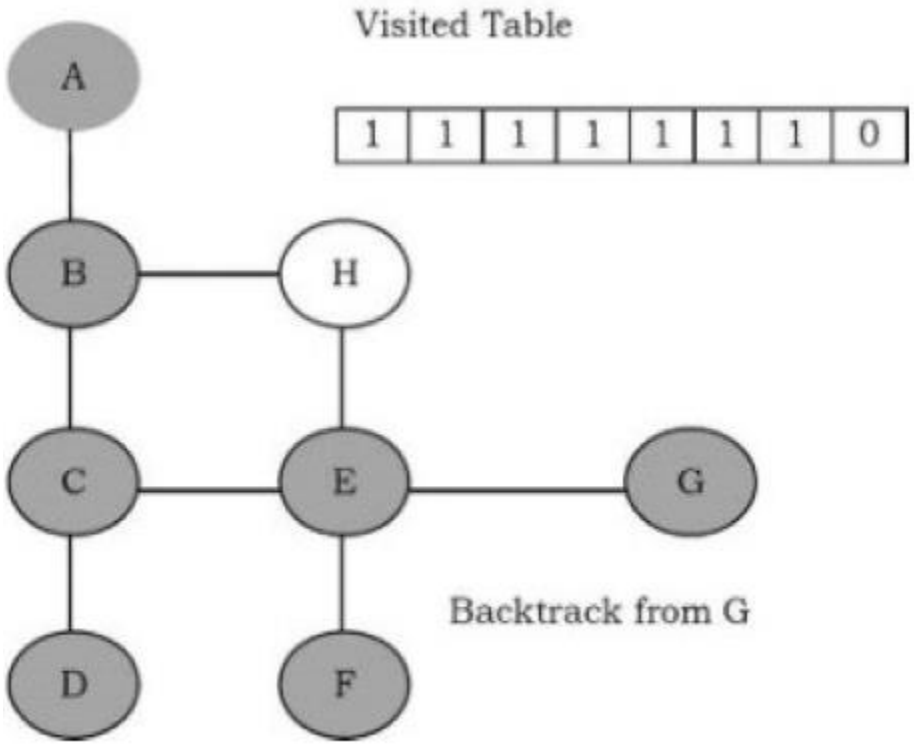

A
Visited Table
1 1 1 0 0 0 0 0
B
H
C
E
G
Recursive call of DFS, vertex C is visited
D
F

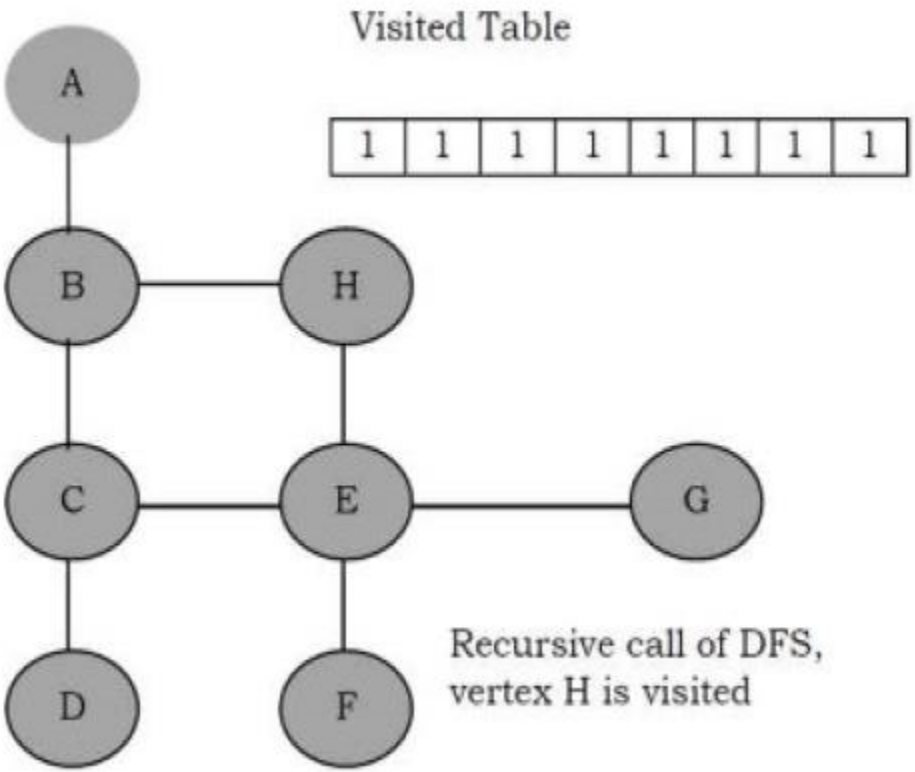

A
Visited Table
1 1 1 1 0 0 0 0
B
H
C
E
G
Recursive call of DFS,
vertex D is visited
D
F

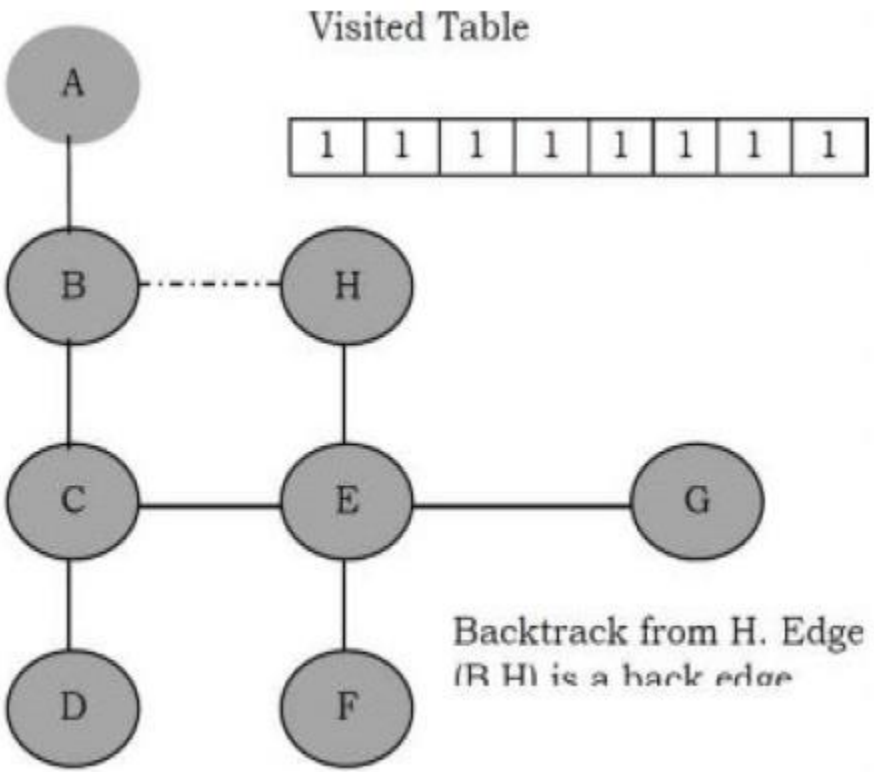

Visited Table
A
1 1 1 1 1 0 0 0
B
H
C
E
G
D
F
Recursive call of DFS, vertex
E is visited

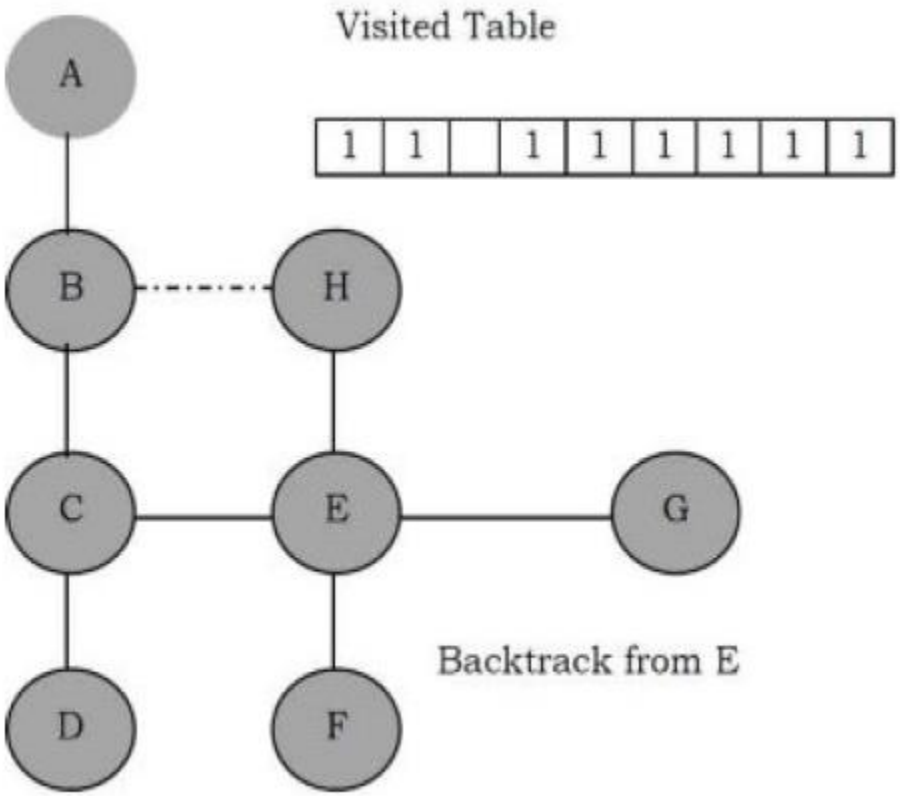

A
Visited Table
1 1 1 1 1 1 0 0
B
H
C
E
G
D
F
Recursive call of DFS,
vertex F is visited

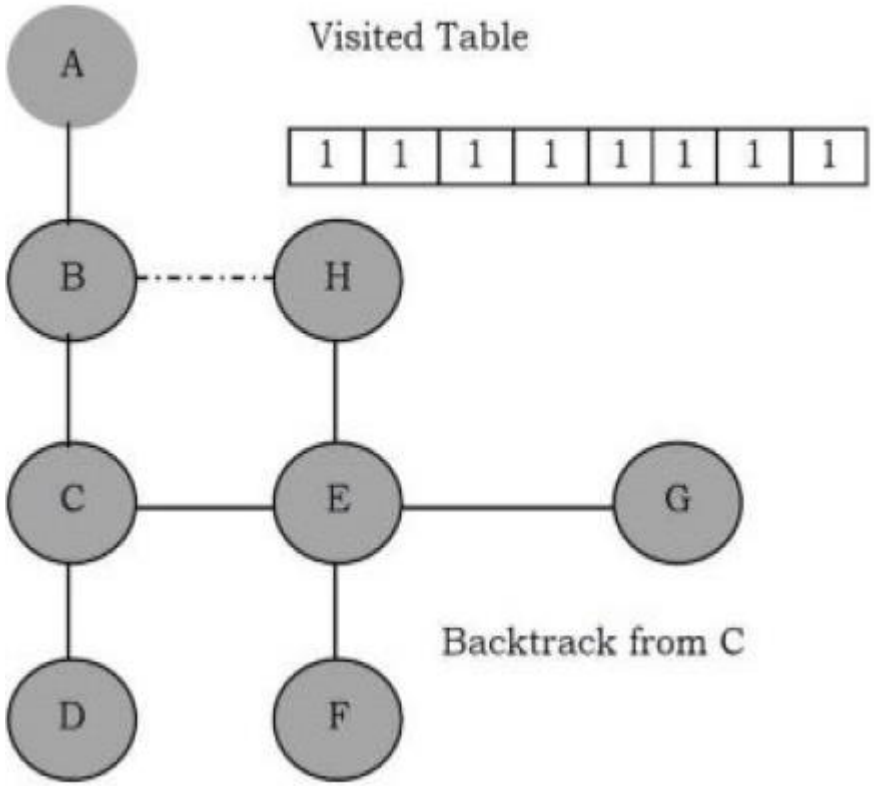

A
Visited Table
1 1 1 1 1 1 0 0
B
H
C
E
G
Backtrack from F
D
F

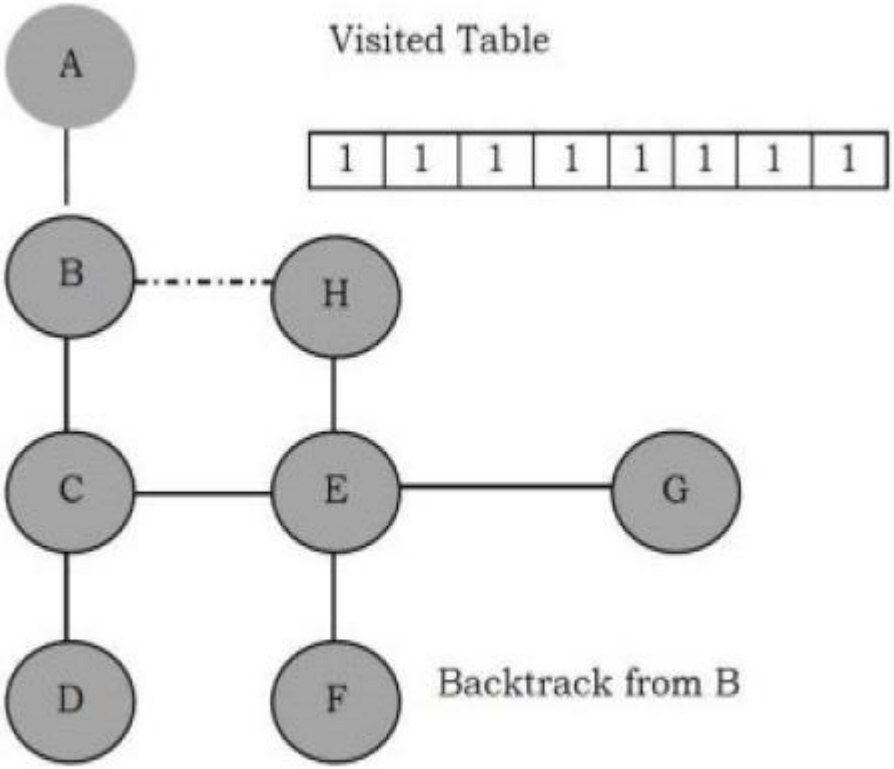

Visited Table
A
1 1 1 1 1 1 1 0
B
H
C
E
G
Backtrack from G
D
F

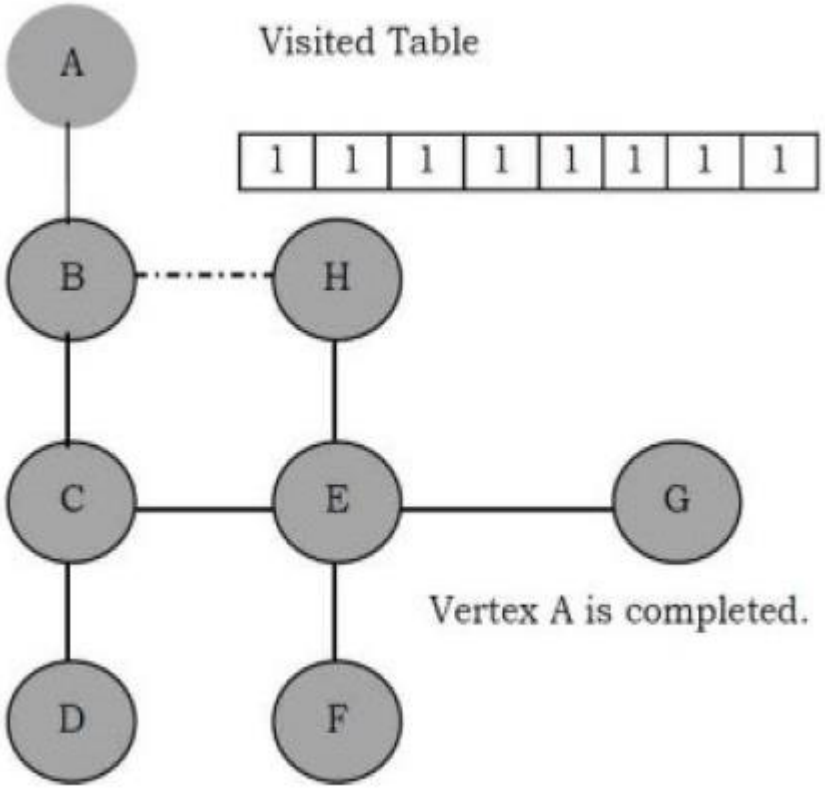

Visited Table
A
1 1 1 1 1 1 1 1
B
H
C
E
G
D
F
Recursive call of DFS,
vertex H is visited


Visited Table
A
1 1 1 1 1 1 1 1
B
H
C
E
G
D
F
Backtrack from H. Edge
(B H) is a back edge


Visited Table
A
1 1 1 1 1 1 1 1 1
B
H
C
E
G
Backtrack from E
D
F


Visited Table
A
1 1 1 1 1 1 1 1
B
H
C
E
G
Backtrack from C
D
F


A
Visited Table
1 1 1 1 1 1 1 1
B
H
C
E
G
D
F
Backtrack from B


A
Visited Table
1 1 1 1 1 1 1 1
B
H
C
E
G
Vertex A is completed.
D
F

- From the above diagrams, it can be seen that the DFS traversal creates a tree (without back edges) and we call such tree a *DFS tree*. The above algorithm works even if the given graph has connected components.
- The time complexity of DFS is O(*V + E*), if we use adjacency lists for representing the graphs. This is because we are starting at a vertex and processing the adjacent nodes only if they are not visited. Similarly, if an adjacency matrix is used for a graph representation, then all edges adjacent to a vertex can't be found efficiently, and this gives O(*V*2) complexity.

- **Applications of DFS**
  - Topological sorting
  - Finding connected components
  - Finding articulation points (cut vertices) of the graph
  - Finding strongly connected components
  - Solving puzzles such as mazes

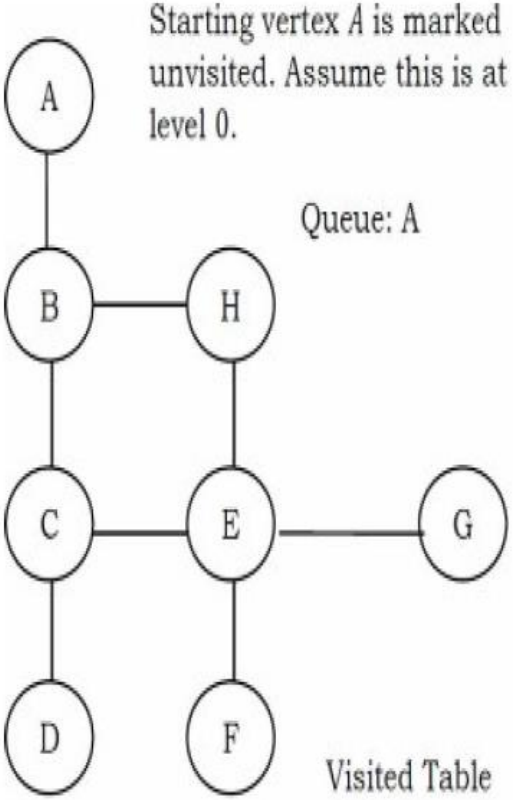
- **Breadth First Search [BFS]:**
- The BFS algorithm works similar to *level – order* traversal of the trees. Like *level – order* traversal, BFS also uses queues. In fact, *level – order* traversal got inspired from BFS. BFS works level by level. Initially, BFS starts at a given vertex, which is at level 0. In the first stage it visits all vertices at level 1 (that means, vertices whose distance is 1 from the start vertex of the graph). In the second stage, it visits all vertices at the second level. These new vertices are the ones which are adjacent to level 1 vertices.

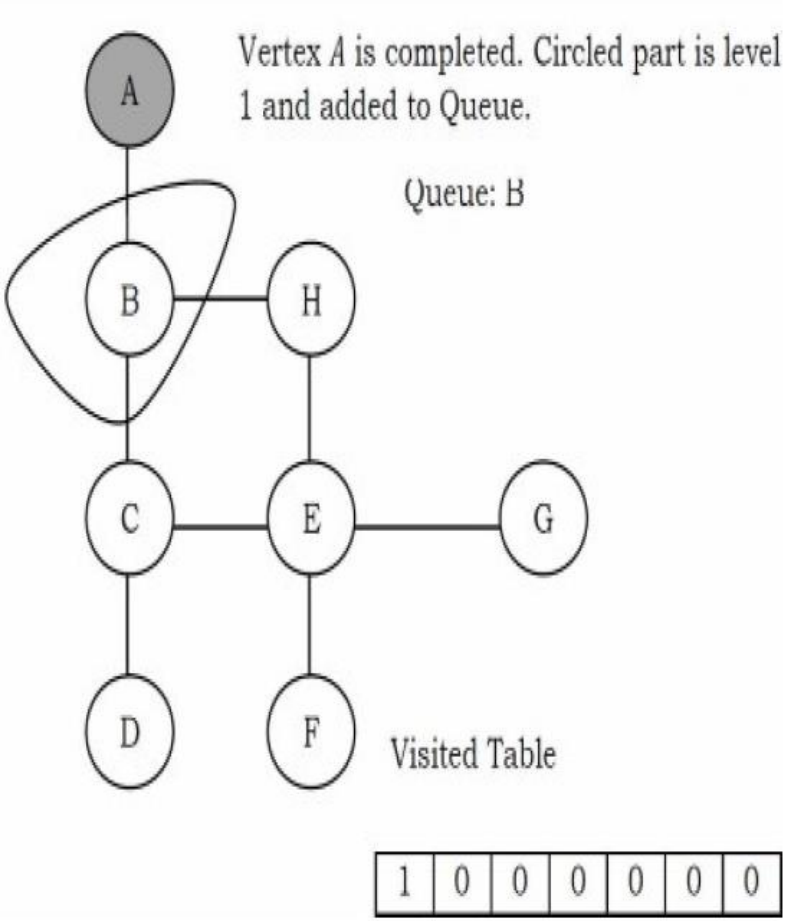
- BFS continues this process until all the levels of the graph are completed. Generally *queue* data structure is used for storing the vertices of a level. As similar to DFS, assume that initially all vertices are marked *unvisited* (*false*). Vertices that have been processed and removed from the queue are marked *visited* (*true*). We use a queue to represent the visited set as it will keep the vertices in the order of when they were first visited. The implementation for the above discussion can be given as:

- **Breadth First Search [BFS]:**
- As similar to DFS, assume that initially all vertices are marked *unvisited* (*false*). Vertices that have been processed and removed from the queue are marked *visited* (*true*).
- We use a queue to represent the visited set as it will keep the vertices in the order of when they were first visited. The implementation for the above discussion can be given as:

```
void BFSTraversal(struct Graph *G) {
    for (int i = 0; i< G→V;i++)
        Visited[i]=0;

    //This loop is required if the graph has more than one component
    for (int i = 0; i< G→V;i++)
        if(!Visited[i])
            BFS(G. i):
}
```

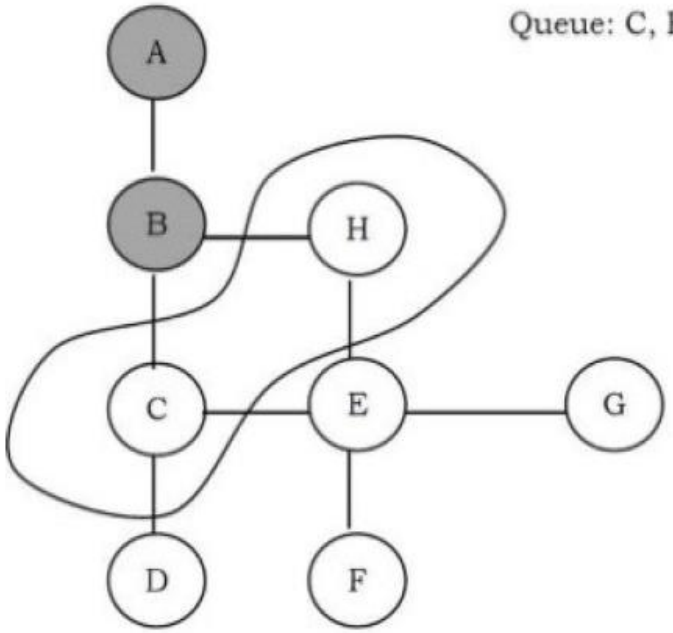
- As an example, let us consider the same graph as that of the DFS example. The BFS traversal can be shown as:


Starting vertex A is marked unvisited. Assume this is at level 0.
A
Queue: A
B
H
C
E
G
D
F
Visited Table
0
0
0
0
0
0
0


Vertex A is completed. Circled part is level 1 and added to Queue.
Queue: B
A
B
H
C
E
G
D
F
Visited Table
1 0 0 0 0 0 0

B is completed. Selected part is level 2 (add to Queue).

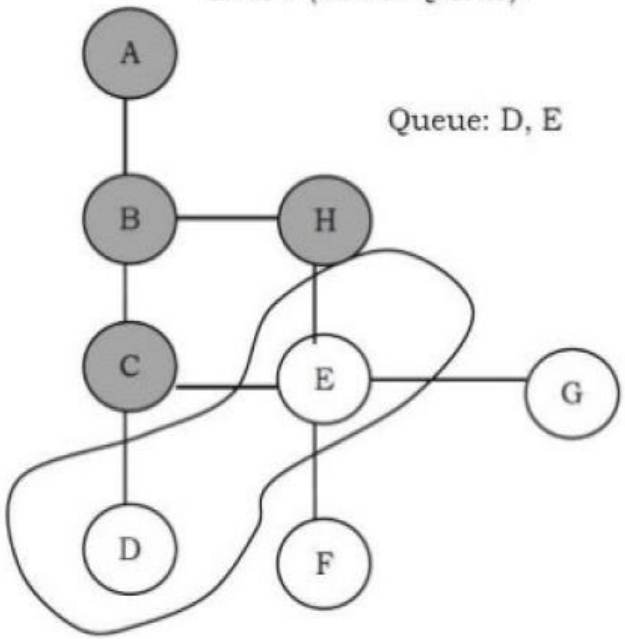
Queue: C, H



Visited Table

1	1	0	0	0	0	0

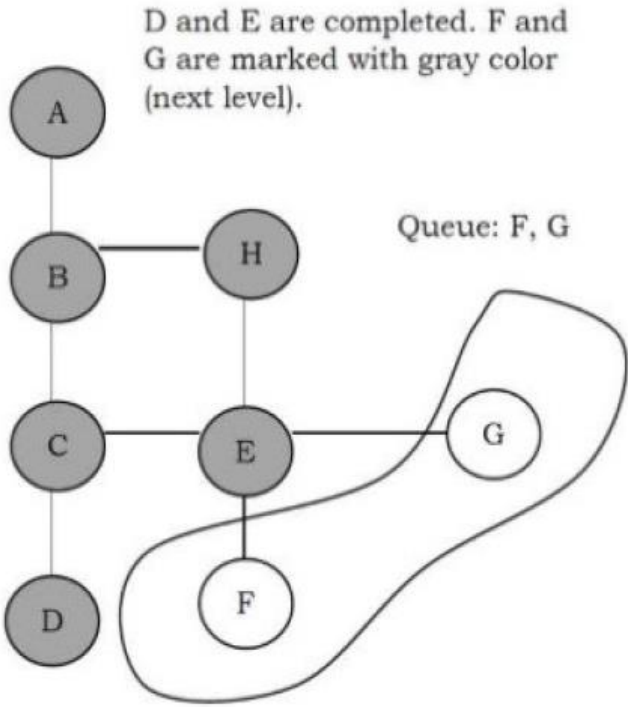
Vertices C and H are completed. Circled part is level 3 (add to Queue).



Queue: D, E

Visited Table

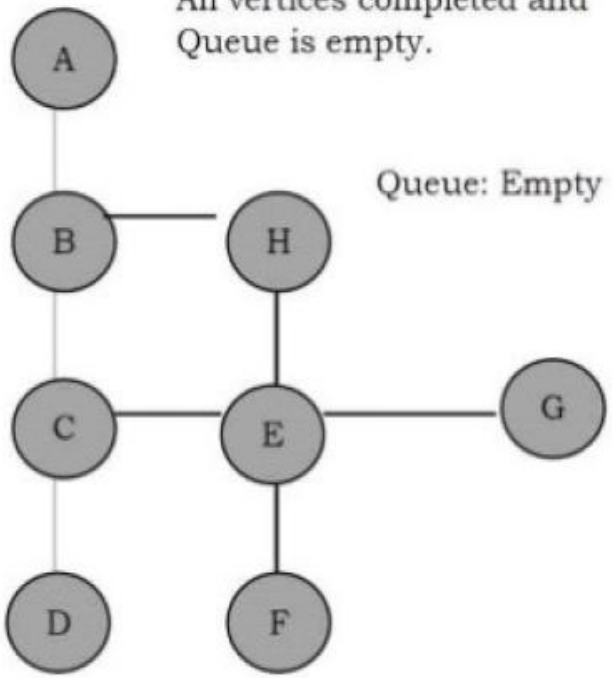
1	1	0	0	0	0	1



Visited Table

1	1	1	1	0	0	1

All vertices completed and
Queue is empty.



Queue: Empty

Visited Table

1	1	1	1	1	1	1

- Time complexity of BFS is O($V + E$), if we use adjacency lists for representing the graphs, and O(V2) for adjacency matrix representation.

- **Applications of BFS**
  - Finding all connected components in a graph
  - Finding all nodes within one connected component
  - Finding the shortest path between two nodes
  - Testing a graph for bipartiteness

- **Comparing DFS and BFS**
- Comparing BFS and DFS, the big advantage of DFS is that it has much lower memory requirements than BFS because it’s not required to store all of the child pointers at each level. Depending on the data and what we are looking for, either DFS or BFS can be advantageous. For example, in a family tree if we are looking for someone who’s still alive and if we assume that person would be at the bottom of the tree, then DFS is a better choice. BFS would take a very long time to reach that last level.

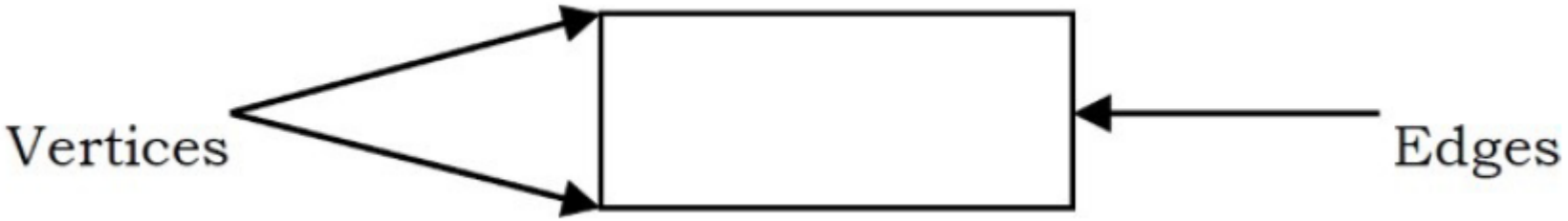
- The DFS algorithm finds the goal faster. Now, if we were looking for a family member who died a very long time ago, then that person would be closer to the top of the tree. In this case, BFS finds faster than DFS. So, the advantages of either vary depending on the data and what we are looking for.
- DFS is related to preorder traversal of a tree. Like *preorder* traversal, DFS visits each node before its children. The BFS algorithm works similar to *level – order* traversal of the trees.

- If someone asks whether DFS is better or BFS is better, the answer depends on the type of the problem that we are trying to solve.
- BFS visits each level one at a time, and if we know the solution we are searching for is at a low depth, then BFS is good. DFS is a better choice if the solution is at maximum depth.
- The below table shows the differences between DFS and BFS in terms of their applications.

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	Yes	Yes
Shortest paths		Yes
Minimal use of memory space	Yes	

Minimal Spanning Tree

- The *Spanning tree* of a graph is a subgraph that contains all the vertices and is also a tree. A graph may have many spanning trees. As an example, consider a graph with 4 vertices as shown below. Let us assume that the corners of the graph are vertices.



- For this simple graph, we can have multiple spanning trees as shown below.
- The algorithm we will discuss now is *minimum spanning tree* in an undirected graph. We assume that the given graphs are weighted graphs. If the graphs are unweighted graphs then we can still use the weighted graph algorithms by treating all weights as equal. A *minimum spanning tree* of an undirected graph *G* is a tree formed from graph edges that connect all the vertices of *G* with minimum total cost (weights). A minimum spanning tree exists only if the graph is connected.
- There are two famous algorithms for this problem:
  - *Prim's* Algorithm
  - *Kruskal's* Algorithm

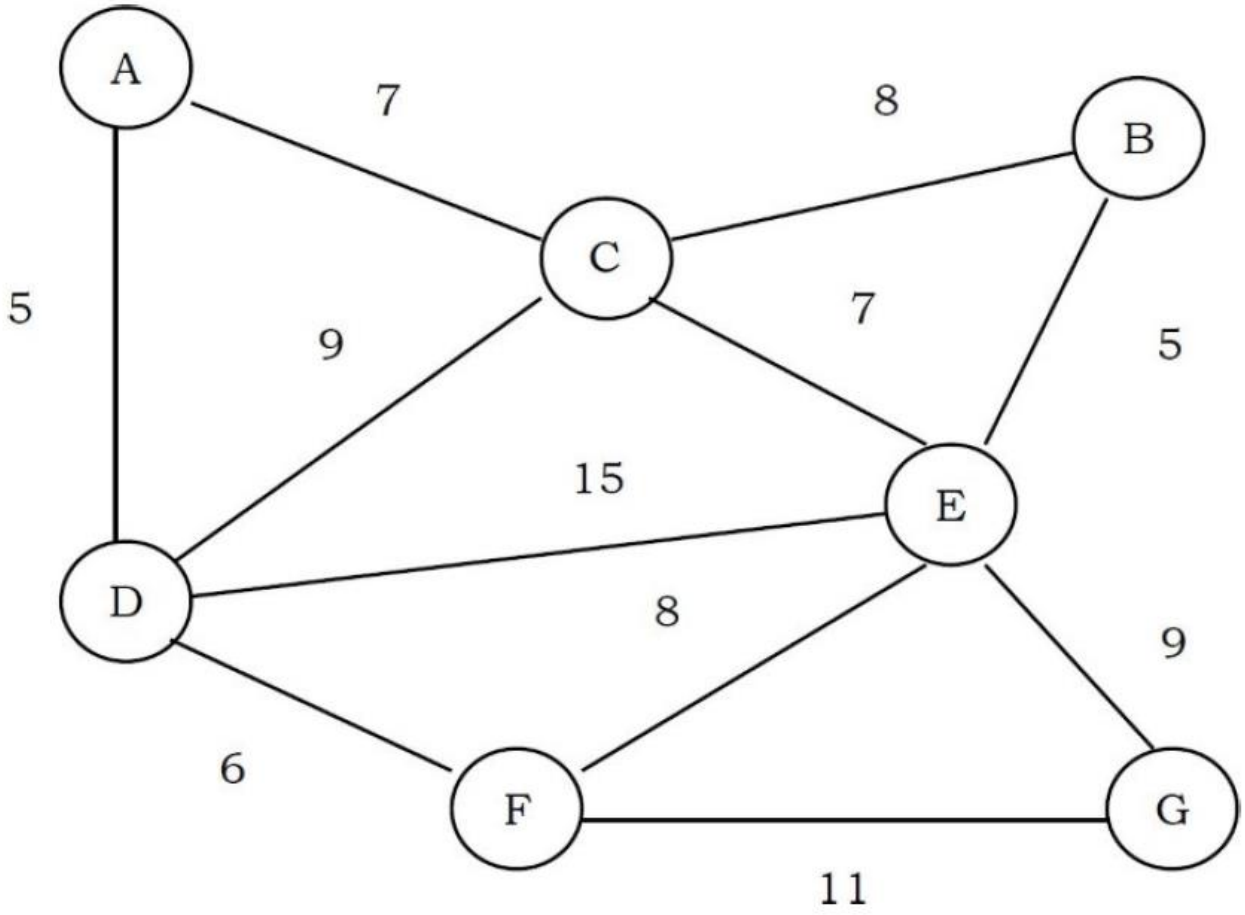
- **Prim's Algorithm:**
- Prim's algorithm is almost the same as Dijkstra's algorithm. As in Dijkstra's algorithm, in Prim's algorithm we keep the values *distance* and *paths* in the distance table. The only exception is that since the definition of *distance* is different, the updating statement also changes a little. The update statement is simpler than before.

```
void Prims(struct Graph *G, int s) {
    struct PriorityQueue *PQ = CreatePriorityQueue();
    int v, w;
    EnQueue(PQ, s);
    Distance[s] = 0;                    // assume the Distance table is filled with -1
    while ((!IsEmptyQueue(PQ)) {
    v = DeleteMin(PQ);
        for all adjacent vertices w of v {
            Compute new distance d= Distance[v] + weight[v][w];
            if(Distance[w] == -1) {
                Distance[w] = weight[v][w];
                Insert w in the priority queue with priority d
                Path[w] = v;
            }
            if(Distance[w]  > new distance d) {
                Distance[w] = weight[v][w];
                Update priority of vertex w to be d;
                Path[w] = v;
            }
        }
    }
}
```

- The entire implementation of this algorithm is identical to that of Dijkstra's algorithm. The running time is O(|*V*|2) without heaps [good for dense graphs], and O (*ElogV*) using binary heaps [good for sparse graphs].

- **Kruskal's Algorithm**
- The algorithm starts with V different trees (V is the vertices in the graph). While constructing the minimum spanning tree, every time Kruskal's alorithm selects an edge that has minimum weight and then adds that edge if it doesn't create a cycle. So, initially, there are | V | single-node trees in the forest. Adding an edge merges two trees into one. When the algorithm is completed, there will be only one tree, and that is the minimum spanning tree. There are two ways of implementing Kruskal's algorithm:
  - By using Disjoint Sets: Using UNION and FIND operations
  - By using Priority Queues: Maintains weights in priority queue

- The appropriate data structure is the UNION/FIND algorithm [for implementing forests]. Two vertices belong to the same set if and only if they are connected in the current spanning forest.
- Each vertex is initially in its own set. If *u* and *v* are in the same set, the edge is rejected because it forms a cycle. Otherwise, the edge is accepted, and a UNION is performed on the two sets containing *u* and *v*.
- As an example, consider the following graph (the edges show the weights).


A
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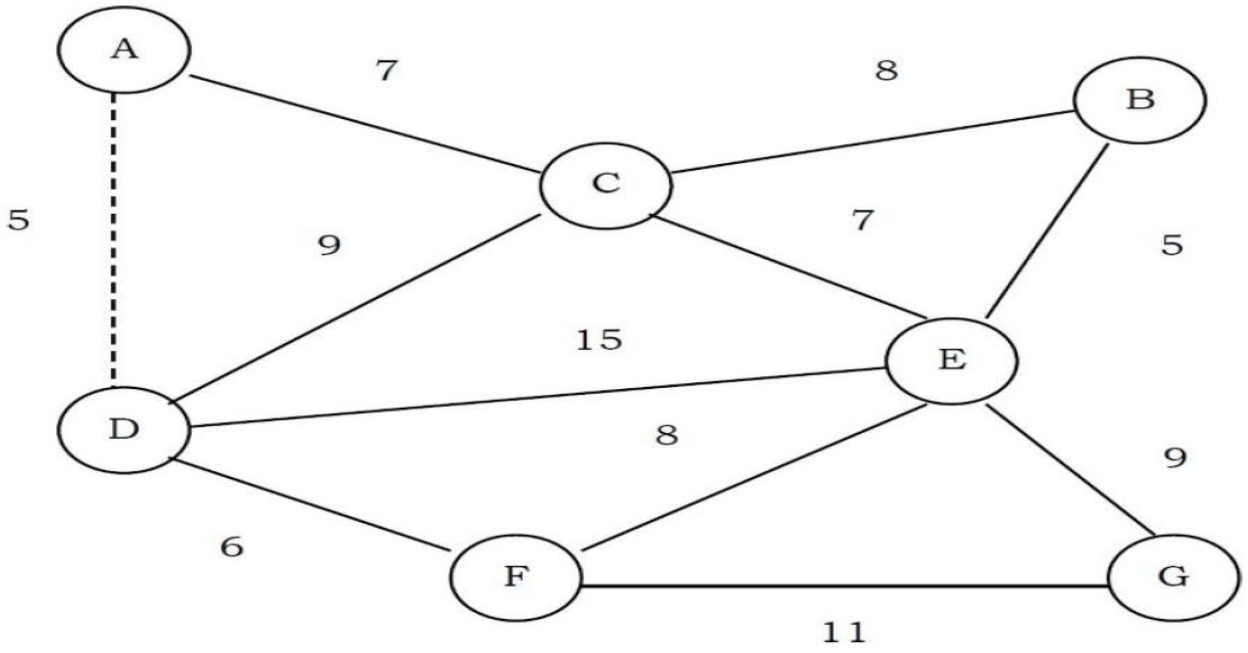
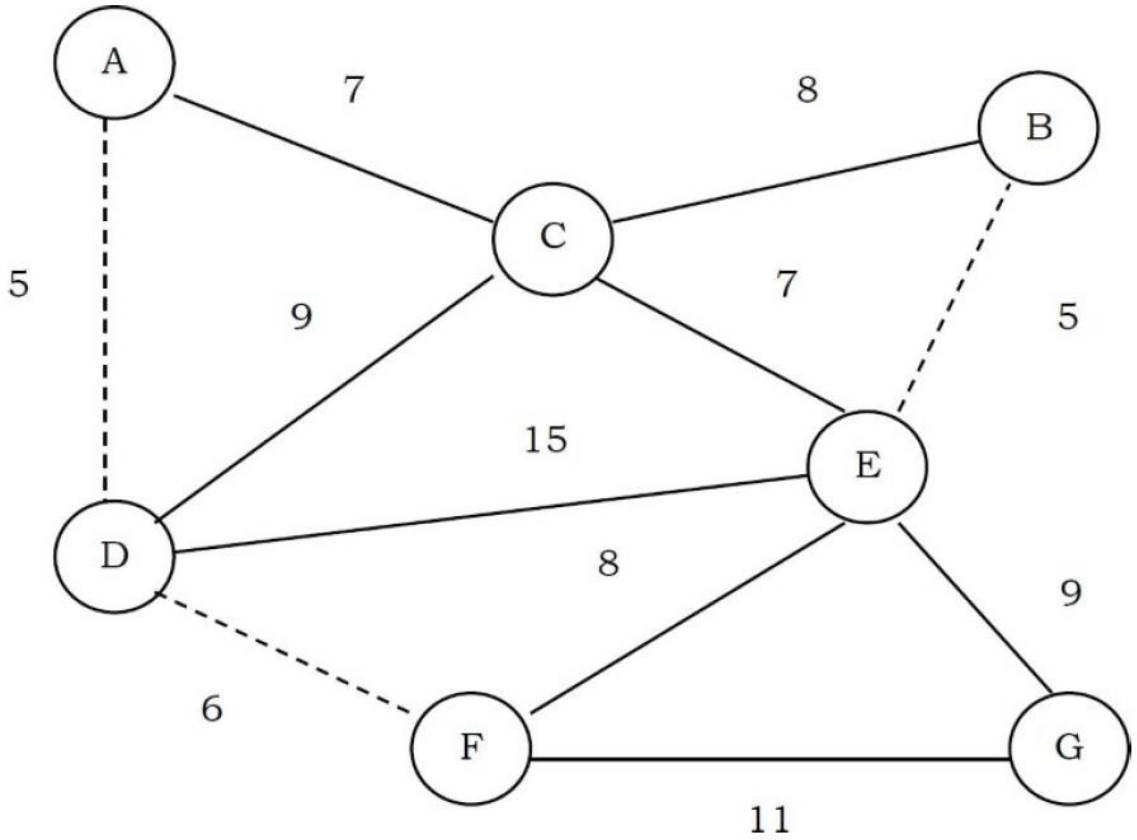
11

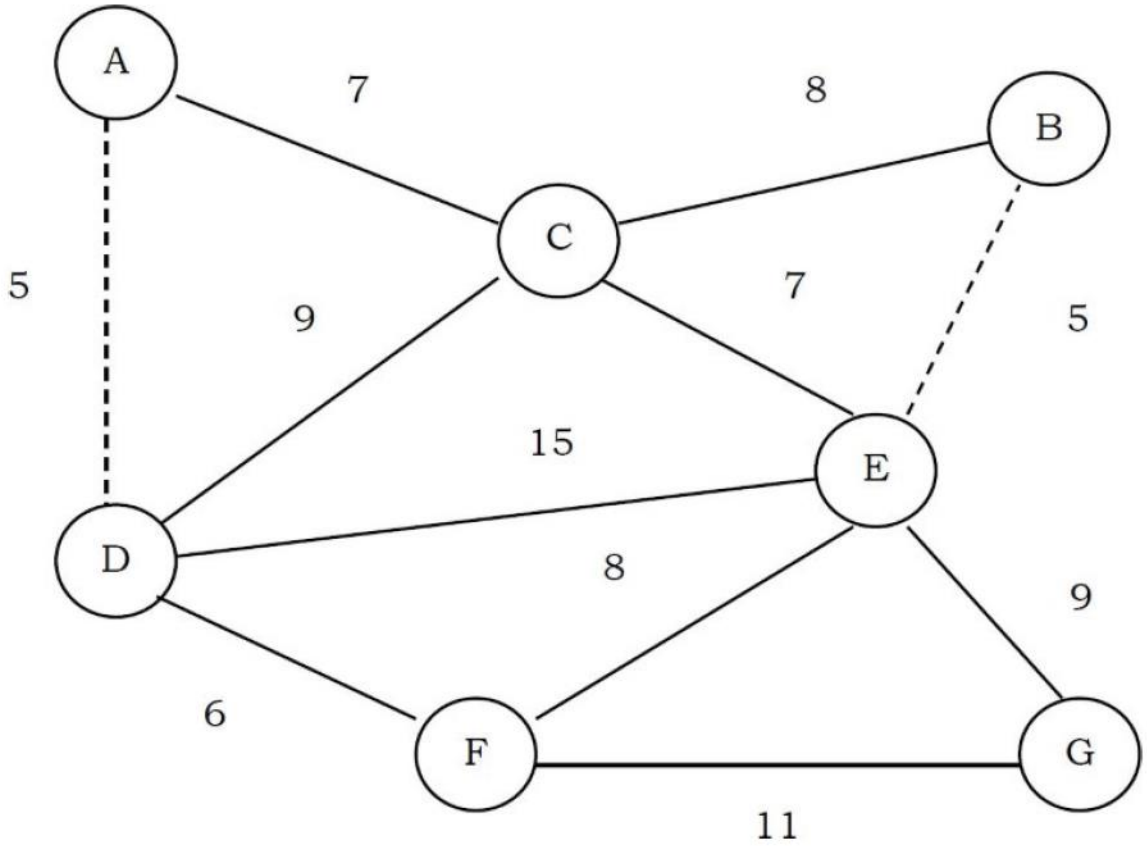
- Now let us perform Kruskal's algorithm on this graph. We always select the edge which has minimum weight.



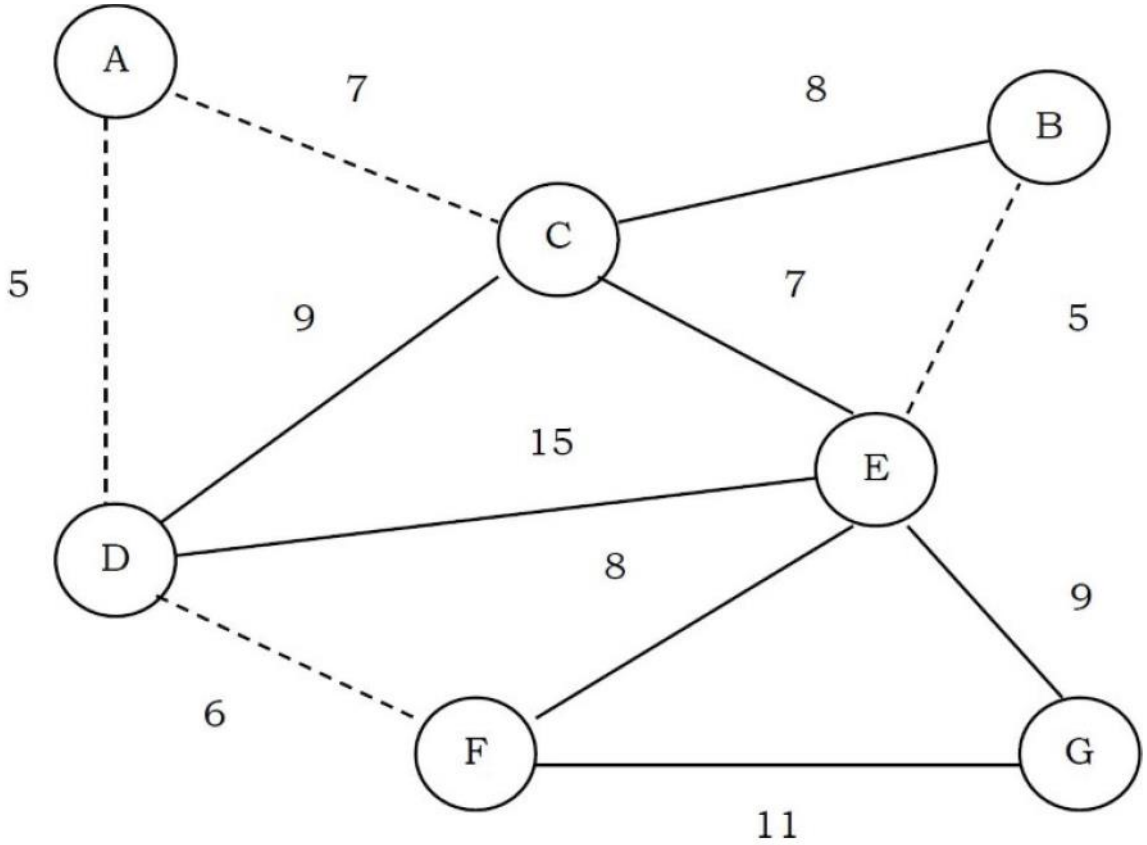
- From the above graph, the edges which have minimum weight (cost) are: AD and BE. From these two we can select one of them and let us assume that we select AD (dotted line).



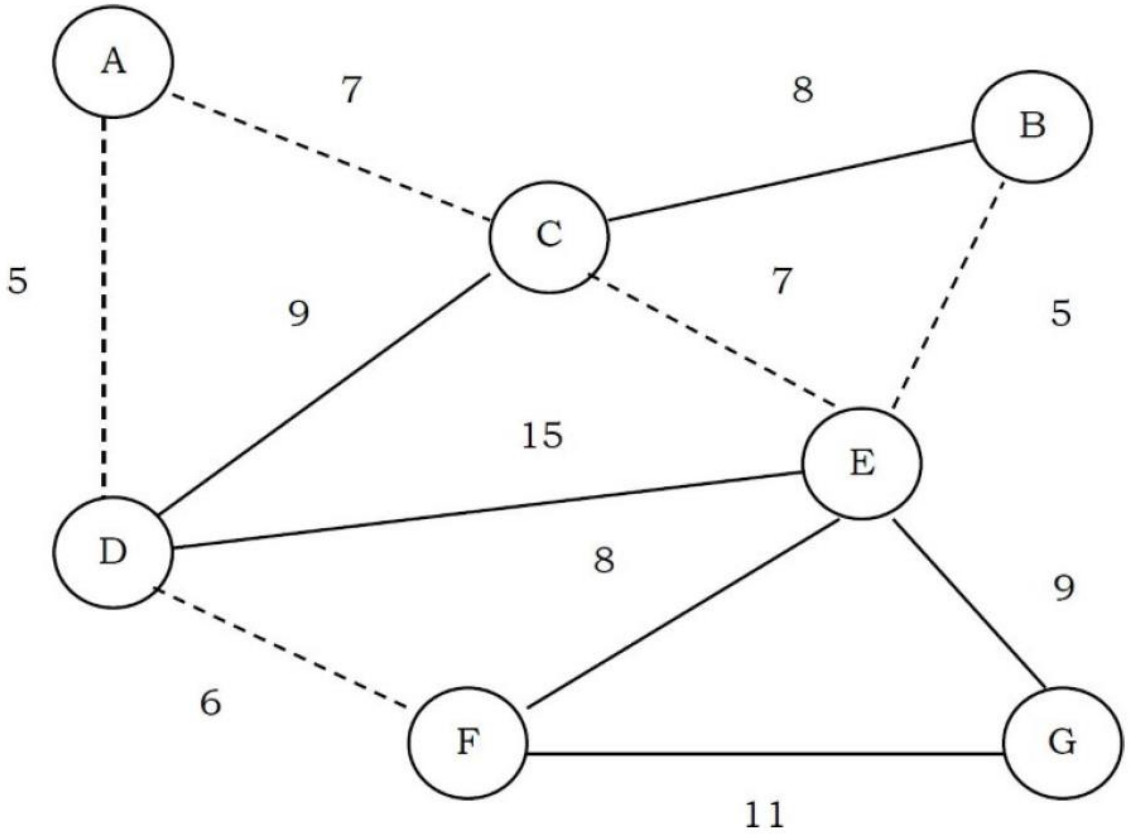
DF is the next edge that has the lowest cost (6).



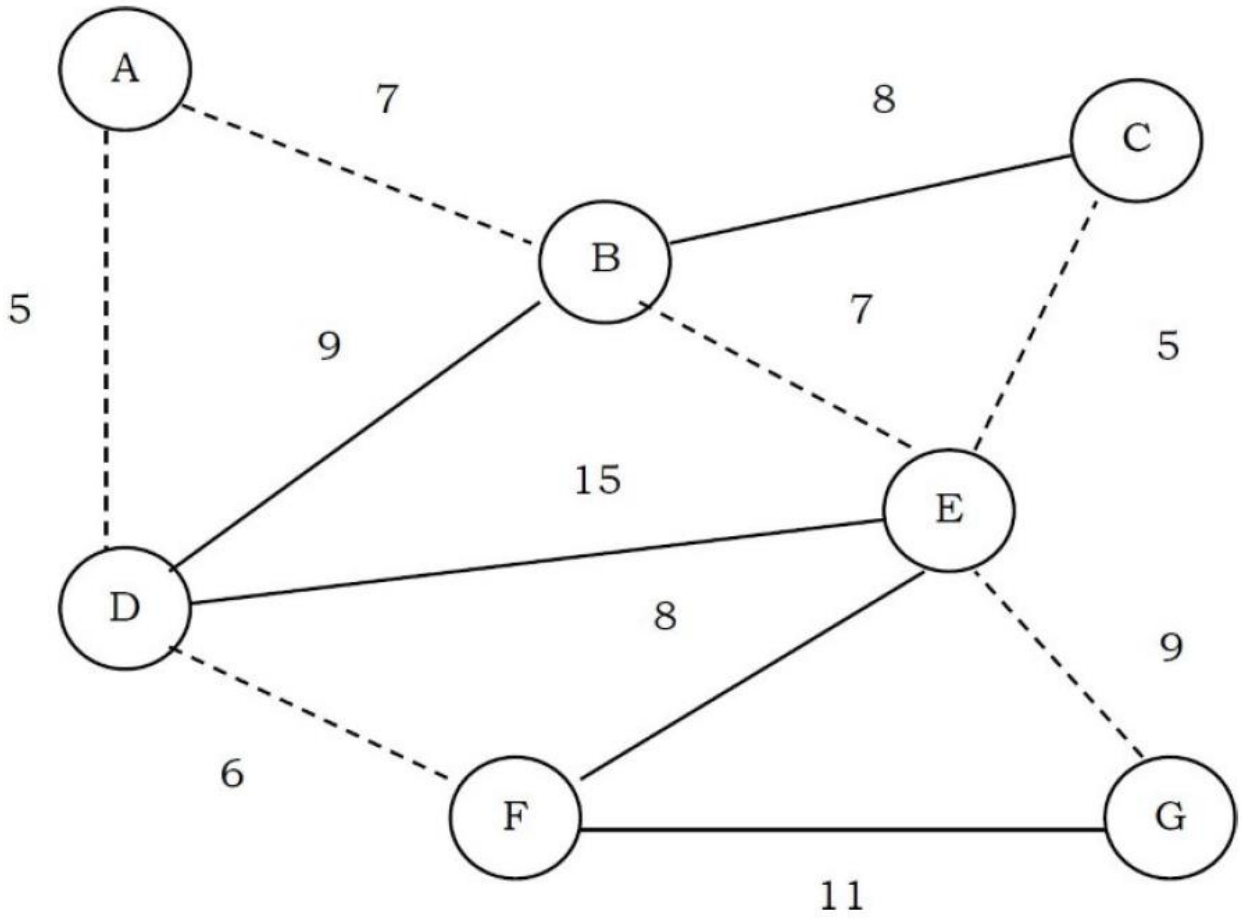
BE now has the lowest cost and we select it (dotted lines indicate selected edges).



Next, AC and CE have the low cost of 7 and we select AC.



Then we select CE as its cost is 7 and it does not form a cycle.


A
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- The next low cost edges are CB and EF. But if we select CB, then it forms a cycle. So we discard it.
- This is also the case with EF. So we should not select those two. And the next low cost is 9 (BD and EG).
- Selecting BD forms a cycle so we discard it. Adding EG will not form a cycle and therefore with this edge we complete all vertices of the graph.

```
void Kruskal(struct Graph *G) {
    S = ϕ;   // At the end S will contains the edges of minimum spanning trees
    for (int v = 0; v< G→V; v++)
        MakeSet (v);
    Sort edges of E by increasing weights w;
    for each edge (u, v) in E { //from sorted list
        if(FIND (u) ≠ FIND (v)) {
            S = S ∪ {(u, v)};
            UNION (u, v);
        }
    }
    return S;
}
```

- **Note:**The worst-case running time of this algorithm is O(*ElogE*), which is dominated by the heap operations.
- That means, since we are constructing the heap with E edges, we need O(*ElogE*) time to do that.